Remainders: notice the cycle, explain the answer

STUDENT COPY · CORE 45–60 MINUTES

Use a separate notebook if you need more space. P1–P4 form the core; P5–P6 are optional extensions. Try before looking at the hints on the final page.

Before you start: divide whole numbers into equal groups, recognise multiples and multiply small integers.

Starting checks

D1

A student writes 43 = 6 × 6 + 7 and says that 43 leaves remainder 7 when divided by 6. The equality is correct. Is the claimed remainder correct? Explain and write the valid division statement.

D2

The colours Red, Blue, Green, Yellow repeat in that order. Position 1 is Red. What colours are in positions 8 and 9? Explain where each position falls in the repeated four-colour blocks.

The rule behind a remainder

For a nonnegative integer N and a positive integer d, division with remainder means writing N = dq + r, where q is a nonnegative integer and 0 ≤ r < d. The number q counts complete groups of size d. The number r counts what remains. Remainder 0 is allowed: it means the groups use everything.

A power means repeated multiplication: $4^2 = 4 \times 4$. For any positive base a, $a^0 = 1$. A marker starts at move 0, but the colour list above starts at position 1.

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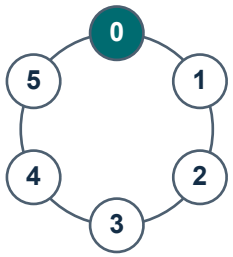
Make a cycle you can justify

A number and its remainder differ by a multiple of the divisor. For example, on division by 7, 18 and 4 have the same remainder because 18 = 7 × 2 + 4. Adding 3 to both keeps their difference a multiple of 7. Multiplying both by 4 also keeps their difference a multiple of 7. This is why we can use small remainders during a long calculation.

To investigate a cycle, record the starting state and apply the same update rule repeatedly. If you return to the starting state, the same rule gives the same next state, then the same following state, and so on. This proves repetition. A few similar-looking entries are only a clue; identify the exact return and the rule that forces the rest.

Worked example E1

Six places around a circle are labelled 0, 1, 2, 3, 4, 5 in clockwise order. A marker starts at 0. Each move advances it exactly four places clockwise, wrapping around the circle. Where is it after 35 moves? Justify the cycle you use.



Labels increase clockwise.

The filled place is the start, after 0 moves.

Trace the moves on your own copy.

1. Record the position after each move, including the start: after 0 moves it is at 0; after 1 move it is at 4; after 2 moves it is at 2; after 3 moves it is back at 0.

2. For the second move, 4 + 4 = 8 leaves remainder 2 on division by 6. For the third, 2 + 4 = 6 leaves remainder 0.

3. The same move rule now starts again from 0. It must produce 4, then 2, then 0 again. Therefore blocks of three moves repeat.

4. Write 35 = 3 × 11 + 2. The first 33 moves are eleven complete cycles and return the marker to 0.

5. The two extra moves take it to 4 and then 2. Its position after 35 moves is 2.

Explain it to a partner

What is the position after 36 moves? Point to the repeated state and say why the same moves must repeat. Compare this move-0 start with the position-1 colour list.

Powers: keep both divisions clear

E2 · Track a power by its remainder

Find the remainder of 4^{20} on division by 7. Then decide whether $4^{21} - 1$ is divisible by 7. Explain the repeated pattern rather than calculating the large powers.

1. Begin at exponent 0. The value $4^0 = 1$ has remainder 1. Each increase in the exponent multiplies the preceding power by 4.

2. Starting with remainder 1, the next remainder is 4. Multiplying 4 by 4 gives 16, with remainder 2. Multiplying 2 by 4 gives 8, with remainder 1.

3. We have returned to remainder 1 after three multiplications. Repeating the same multiply-by-4 rule gives the cycle 1, 4, 2, indexed by exponents with remainders 0, 1, 2 on division by 3.

4. Since $20 = 3 \times 6 + 2$, the exponent has remainder 2 on division by 3. The corresponding remainder of 4^{20} on division by 7 is 2.

5. Since $21 = 3 \times 7$, the power 4^{21} has remainder 1 on division by 7. Subtracting 1 leaves remainder 0. Thus $4^{21} - 1$ is divisible by 7.

6. As a check on the reason for the cycle, $4^3 = 64 = 7 \times 9 + 1$. Multiplying by another block of 4^3 leaves a remainder unchanged.

Exponent k	0	1	2	3
Remainder of 4^k on division by 7	1	4	2	1

Exponents repeat their position every 3 steps; the power remainders are calculated on division by 7.

Why small remainders are enough

The same idea works for a product. Let A and B be nonnegative integers and let d be a positive divisor. Write $A = dq + r$ and $B = dt + s$. Here q and t count complete groups of size d; r and s are the respective remainders. Multiplying the two expressions gives $AB = d^2qt + dqs + drt + rs = d(dqt + qs + rt) + rs$. Each of the first three terms contains a factor d. Therefore AB and rs differ by a multiple of d and have the same remainder. If rs is at least d, divide rs once more to get a remainder smaller than d. We may add and multiply remainders this way; this pack does not use division of remainders.

A complete remainder argument considers every permitted remainder. On division by d, the list 0, 1, …, $d - 1$ covers every integer in this pack. A table using all those cases can prove that a proposed remainder is impossible; a table containing only a few chosen input values cannot do that.

Core practice · P1–P4

Show the calculation and explain why your cycle or list is complete. Hints are on page 6.

P1 · Core 1 — make the list complete

Find all integers N with 30 ≤ N ≤ 60 that leave remainder 5 when divided by 8. Explain why your list is complete.

P2 · Core 1 — track the start and reachable places

Eight places around a circle are labelled 0, 1, 2, 3, 4, 5, 6, 7 in clockwise order. A marker starts at 3 and advances exactly six places clockwise on every move. Find its position after 38 moves. Also list all places it can ever occupy, including the starting place, and justify that list.

P3 · Core 2 — use all remainder cases

For a nonnegative integer n, which remainders can n(n + 2) leave when divided by 3? Give an example for each possible remainder and prove that your list covers every n.

P4 · Core 2 — reverse a cycle question

Among the integers k with 0 ≤ k ≤ 14, find every k for which 4^k leaves remainder 2 on division by 7. How many such exponents are there?

Extensions and a final check

One useful change of representation appears in Exercise 5. Appending digit 2 to a whole number N produces 10N + 2. For example, appending 2 to 22 gives 10 × 22 + 2 = 222. To track its remainder, multiply the old remainder by 10, add 2, and divide by the chosen divisor again.

P5 · Extension — discover the update rule

For a positive integer k, let N_k be the decimal integer made of exactly k copies of the digit 2. Thus $N_1 = 2$, $N_2 = 22$ and $N_3 = 222$. Find all k with $1 \le k \le 30$ for which N_k is divisible by 7. Prove that you have not missed a value of k.

P6 · Proof extension — combine two cycles

Find all nonnegative integers n for which $4^n + 2^n + 1$ is divisible by 7. Prove that your description is necessary and sufficient.

X1

Start with t = 0. On each move replace t by the remainder of 3t + 1 on division by 5, so the displayed value is always one of 0, 1, 2, 3, 4. What is displayed after 100 moves? Explain why the cycle you use must repeat.

Hints, then your next step

P1

Hint 1: Write a number with the required remainder as $8q + 5$, where q is a nonnegative integer.

Hint 2: Find the last such number below 30, then add 8 until you pass 60.

P2

Hint 1: Write the positions after 0, 1, 2, 3 and 4 moves. Reduce each position to a remainder from 0 to 7.

Hint 2: Once the marker returns to 3, the same moves repeat. Divide 38 by the number of moves in this cycle.

P3

Hint 1: The remainder of n on division by 3 must be 0, 1 or 2. Treat these as three cases.

Hint 2: For each remainder r, calculate $r(r + 2)$ and reduce that result on division by 3. Explain why replacing n by r is allowed.

P4

Hint 1: Use the power cycle established in Example E2, including the starting exponent 0.

Hint 2: Remainder 2 occurs at exponents of the form $3q + 2$. List these inside the given interval.

P5

Hint 1: Appending one digit 2 gives $N_{k+1} = 10N_k + 2$. Track only the remainder on division by 7.

Hint 2: For convenient starting notation set $N_0 = 0$. From remainder r, the next remainder is that of $10r + 2$. Continue until you return to remainder 0.

P6

Hint 1: You already know the remainders of 4^n. Make a second list for 2^n, starting at $n = 0$.

Hint 2: Both lists repeat after three exponent steps. Add their remainders and 1 separately for n having remainder 0, 1 or 2 on division by 3.

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