Remainders · Teacher guide

Remainders: notice the cycle, explain the answer

Use with the separate six-page student copy. Original skill development; not a national mock or a contest-readiness assessment.

A flexible 58-minute core lesson

5 minutes — D1–D2: diagnose valid remainders and cycle indexing. Ask for the reason as well as the answer.

6 minutes — Teach the division statement and why addition or multiplication can use remainders. Keep the symbolic product expansion available for learners who need the proof.

10 minutes — Model E1 and E2. Ask a learner to identify the exact repeated state and the rule that forces repetition.

16 minutes — Learners attempt P1–P4. Give Hint 1 first, allow another attempt, then offer Hint 2. Pair discussion should focus on why the answer list or cycle is complete.

8 minutes — Use the paper-to-practice routine on the existing lesson. Treat this as a flexible allowance; move it to follow-up if learners need longer.

5 minutes — Retry one original paper question, then write which step changed in the learner's explanation.

8 minutes — Complete X1 and review its cycle argument. Use any remaining time to introduce P5 or P6 as a follow-up extension.

For 45 minutes

Use D1–D2, the two examples and P1–P3 in class. Use P4 as a quick oral transfer check if time allows. Keep X1 as the final written check and move the online session plus P5–P6 to follow-up. Do not rush the starting-index explanation to finish every page.

Keep the learning evidence useful

Use the self-check rubric for one selected written answer. Record whether the explanation was independent, hint-supported or solution-assisted. These classroom notes are not requested as website analytics.

Keep hints hidden until needed. Let partners compare the first point where their reasoning differs. Invite a student to explain the repeated-state argument to a friend, rather than merely exchange final answers.

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Theory and prerequisite bridges

For a nonnegative integer N and a positive integer d, division with remainder means writing $N = dq + r$, where q is a nonnegative integer and $0 \le r < d$. The number q counts complete groups of size d. The number r counts what remains. Remainder 0 is allowed: it means the groups use everything.

A number and its remainder differ by a multiple of the divisor. For example, on division by 7, 18 and 4 have the same remainder because $18 = 7 \times 2 + 4$. Adding 3 to both keeps their difference a multiple of 7. Multiplying both by 4 also keeps their difference a multiple of 7. This is why we can use small remainders during a long calculation.

The same idea works for a product. Let A and B be nonnegative integers and let d be a positive divisor. Write $A = dq + r$ and $B = dt + s$. Here q and t count complete groups of size d; r and s are the respective remainders. Multiplying the two expressions gives $AB = d^2qt + dqs + drt + rs = d(dqt + qs + rt) + rs$. Each of the first three terms contains a factor d. Therefore AB and rs differ by a multiple of d and have the same remainder. If rs is at least d, divide rs once more to get a remainder smaller than d. We may add and multiply remainders this way; this pack does not use division of remainders.

To investigate a cycle, record the starting state and apply the same update rule repeatedly. If you return to the starting state, the same rule gives the same next state, then the same following state, and so on. This proves repetition. A few similar-looking entries are only a clue; identify the exact return and the rule that forces the rest.

Keep the index visible. A marker's starting position is its position after 0 moves. Powers begin here at exponent 0, with $a^0 = 1$ for positive a. The colours in D2 were instead numbered from position 1. A zero remainder in the step count means complete cycles with no extra moves, but its interpretation depends on where the numbering starts.

A complete remainder argument considers every permitted remainder. On division by d, the list $0, 1, \ldots, d - 1$ covers every integer in this pack. A table using all those cases can prove that a proposed remainder is impossible; a table containing only a few chosen input values cannot do that.

One useful change of representation appears in Exercise 5. Appending digit 2 to a whole number N produces $10N + 2$. For example, appending 2 to 22 gives $10 \times 22 + 2 = 222$. To track its remainder, multiply the old remainder by 10, add 2, and divide by the chosen divisor again.

If a learner needs a bridge

If D1 fails, use counters or small drawn groups before algebraic notation. Ask the learner to identify a full group inside the claimed remainder.

If D2 fails, number a physical or drawn colour strip. Compare a list beginning at position 1 with a marker sequence whose start is move 0.

For E1 and P2, draw a circle with the stated labels and trace only one complete cycle. No geometry measurement is involved.

Starting checks and the first example

D1

A student writes 43 = 6 × 6 + 7 and says that 43 leaves remainder 7 when divided by 6. The equality is correct. Is the claimed remainder correct? Explain and write the valid division statement.

1. The equality is correct, but a remainder on division by 6 must be one of 0, 1, 2, 3, 4 or 5.

2. The seven leftover counters contain another full group of six, with one counter left.

3. Therefore 43 = 6 × 7 + 1. The quotient is 7 and the remainder is 1.

If uncertain: Draw six groups of six and seven loose counters. Move six loose counters into one more group, then retry the question.

D2

The colours Red, Blue, Green, Yellow repeat in that order. Position 1 is Red. What colours are in positions 8 and 9? Explain where each position falls in the repeated four-colour blocks.

1. Positions 1–4 form the first block and positions 5–8 form the second block.

2. Position 8 is the fourth position of a complete block, so it is Yellow.

3. Position 9 starts the next block, so it is Red. A count divisible by 4 ends a complete block; it does not mean there is a colour in position 0.

If uncertain: Write two full blocks with positions 1–8 underneath. Circle the last entry of each block before reducing a larger position number.

E1 · Follow a short cycle instead of making 35 moves

Six places around a circle are labelled 0, 1, 2, 3, 4, 5 in clockwise order. A marker starts at 0. Each move advances it exactly four places clockwise, wrapping around the circle. Where is it after 35 moves? Justify the cycle you use.

1. Record the position after each move, including the start: after 0 moves it is at 0; after 1 move it is at 4; after 2 moves it is at 2; after 3 moves it is back at 0.

2. For the second move, 4 + 4 = 8 leaves remainder 2 on division by 6. For the third, 2 + 4 = 6 leaves remainder 0.

3. The same move rule now starts again from 0. It must produce 4, then 2, then 0 again. Therefore blocks of three moves repeat.

4. Write 35 = 3 × 11 + 2. The first 33 moves are eleven complete cycles and return the marker to 0.

5. The two extra moves take it to 4 and then 2. Its position after 35 moves is 2.

Ask what happens after 36 moves before revealing the answer: twelve complete cycles leave the marker at its starting position 0. Exercise 2 changes the start and asks which places can ever be reached.

Power cycle and complete lists

E2 · Track a power by its remainder

Find the remainder of 4^{20} on division by 7. Then decide whether $4^{21} - 1$ is divisible by 7. Explain the repeated pattern rather than calculating the large powers.

1. Begin at exponent 0. The value $4^0 = 1$ has remainder 1. Each increase in the exponent multiplies the preceding power by 4.

2. Starting with remainder 1, the next remainder is 4. Multiplying 4 by 4 gives 16, with remainder 2. Multiplying 2 by 4 gives 8, with remainder 1.

3. We have returned to remainder 1 after three multiplications. Repeating the same multiply-by-4 rule gives the cycle 1, 4, 2, indexed by exponents with remainders 0, 1, 2 on division by 3.

4. Since $20 = 3 \times 6 + 2$, the exponent has remainder 2 on division by 3. The corresponding remainder of 4^{20} on division by 7 is 2.

5. Since $21 = 3 \times 7$, the power 4^{21} has remainder 1 on division by 7. Subtracting 1 leaves remainder 0. Thus $4^{21} - 1$ is divisible by 7.

6. As a check on the reason for the cycle, $4^3 = 64 = 7 \times 9 + 1$. Multiplying by another block of 4^3 leaves a remainder unchanged.

Exponent k	0	1	2	3
Remainder of 4^k on division by 7	1	4	2	1

Exponents repeat their position every 3 steps; the power remainders are calculated on division by 7.

P1 · Core 1 — make the list complete

Find all integers N with $30 \le N \le 60$ that leave remainder 5 when divided by 8. Explain why your list is complete.

1. Every permitted number has the form $N = 8q + 5$.

2. For $q = 3$ this is 29, which is below the allowed interval. For $q = 4$, 5 and 6 it gives 37, 45 and 53.

3. For every $q \le 3$, the value $8q + 5$ is at most 29, below the interval. For every $q \ge 7$, it is at least 61, above the interval. Thus only $q = 4, 5, 6$ are possible.

4. Therefore the complete list is 37, 45, 53. Checking division gives $37 = 8 \times 4 + 5$, $45 = 8 \times 5 + 5$ and $53 = 8 \times 6 + 5$.

Reachable places and complete cases

P2 · Core 1 — track the start and reachable places

Eight places around a circle are labelled 0, 1, 2, 3, 4, 5, 6, 7 in clockwise order. A marker starts at 3 and advances exactly six places clockwise on every move. Find its position after 38 moves. Also list all places it can ever occupy, including the starting place, and justify that list.

1. The positions after 0, 1, 2, 3 and 4 moves are 3, 1, 7, 5, 3. For instance, the first move gives 3 + 6 = 9, with remainder 1 on division by 8.

2. After four moves we return to the starting place. Because the update rule is unchanged, the four-move block repeats forever.

3. Since 38 = 4 × 9 + 2, nine complete cycles are followed by two extra moves. Starting again at 3, those moves go to 1 and then 7. The requested position is 7.

4. The only occupied places are 1, 3, 5 and 7. Each appears in the first cycle, and every later move repeats a place in that cycle. This establishes both that all four occur and that no other place can occur.

P3 · Core 2 — use all remainder cases

For a nonnegative integer n, which remainders can n(n + 2) leave when divided by 3? Give an example for each possible remainder and prove that your list covers every n.

1. Let r be the remainder of n on division by 3. The possibilities r = 0, 1, 2 cover every nonnegative integer n.

2. The numbers n and r differ by a multiple of 3. Hence n + 2 and r + 2 also differ by a multiple of 3. The product rule in the theory shows that n(n + 2) and r(r + 2) have the same remainder.

3. If r = 0, then r(r + 2) = 0, with remainder 0. If r = 1, the product is 3, again with remainder 0. If r = 2, the product is 8, with remainder 2.

4. Thus the only possible remainders are 0 and 2. Both occur: n = 0 gives product 0, and n = 2 gives product 8.

5. Remainder 1 is impossible because all three possible remainder cases for n have been examined.

Reverse the question; build an update

P4 · Core 2 — reverse a cycle question

Among the integers k with $0 \le k \le 14$, find every k for which 4^k leaves remainder 2 on division by 7. How many such exponents are there?

1. Example E2 proves that the remainders of 4^k on division by 7 are 1, 4, 2 for exponent remainders 0, 1, 2 on division by 3.

2. Therefore the required exponents are exactly those with remainder 2 on division by 3: $k = 3q + 2$.

3. Within $0 \le k \le 14$, the choices are 2, 5, 8, 11 and 14, obtained from $q = 0, 1, 2, 3, 4$.

4. The preceding value would be -1 and the next would be 17, both outside the interval. Thus the list is complete and contains five exponents. Exponent 0 is included in the search interval but does not work, since $4^0 = 1$.

P5 · Extension — discover the update rule

For a positive integer k, let N_k be the decimal integer made of exactly k copies of the digit 2. Thus $N_1 = 2$, $N_2 = 22$ and $N_3 = 222$. Find all k with $1 \le k \le 30$ for which N_k is divisible by 7. Prove that you have not missed a value of k.

1. Use $N_0 = 0$ only as a starting value for the update. For positive k, N_k has exactly the decimal digits stated in the question.

2. Appending 2 gives $N_{k+1} = 10N_k + 2$. If N_k has remainder r on division by 7, the next number has the same remainder as $10r + 2$, because their difference is ten times a multiple of 7.

3. The remainders at $k = 0, 1, 2, 3, 4, 5, 6$ are 0, 2, 1, 5, 3, 4, 0. The calculations after the start are $2; 22 = 7 \times 3 + 1; 12 = 7 + 5; 52 = 7 \times 7 + 3; 32 = 7 \times 4 + 4$; and $42 = 7 \times 6 + 0$.

4. The remainder has returned to 0 after six updates. The same update rule then repeats exactly the same block. Within that block, remainder 0 occurs only at its end, so a positive k gives remainder 0 exactly when k is a positive multiple of 6.

5. Between 1 and 30 these values are 6, 12, 18, 24 and 30. These and only these give divisibility by 7. The artificial starting index 0 is excluded by the question.

Proof extension and transfer check

P6 · Proof extension — combine two cycles

Find all nonnegative integers n for which $4^n + 2^n + 1$ is divisible by 7. Prove that your description is necessary and sufficient.

1. For 2^n, the remainders on division by 7 begin at 1 when n = 0. Multiplying by 2 gives 2, then 4, then 8 with remainder 1. The same rule therefore repeats the cycle 1, 2, 4.

2. Example E2 gives the cycle 1, 4, 2 for 4^n. Both cycles are indexed by the remainder of n on division by 3.

3. If n has remainder 0 on division by 3, the sum $4^n + 2^n + 1$ has the same remainder as 1 + 1 + 1 = 3. It is not divisible by 7.

4. If n has remainder 1, the corresponding sum of remainders is 4 + 2 + 1 = 7, which has remainder 0. If n has remainder 2, it is 2 + 4 + 1 = 7, again with remainder 0.

5. Every nonnegative integer n lies in exactly one of these three cases. Thus divisibility holds precisely for n = 3q + 1 or n = 3q + 2, where q is any nonnegative integer. This proves both that every listed n works and that no other n works.

6. The boundary case n = 0 belongs to the excluded first case: $4^0 + 2^0 + 1 = 3$.

X1

Start with t = 0. On each move replace t by the remainder of 3t + 1 on division by 5, so the displayed value is always one of 0, 1, 2, 3, 4. What is displayed after 100 moves? Explain why the cycle you use must repeat.

1. Starting at 0, the next value is 1. From 1 the next value is 4. From 4, 3 × 4 + 1 = 13 has remainder 3. From 3, 3 × 3 + 1 = 10 has remainder 0.

2. Thus the values after 0, 1, 2, 3, 4 moves are 0, 1, 4, 3, 0. The same rule has returned to the same starting value, so the four-move block repeats.

3. Since 100 = 4 × 25, there are twenty-five complete blocks and no extra moves. The display is 0. Here index 0 is the starting state, so a multiple of the cycle length returns to that state.

The learner should identify the new multiply-and-add update, justify the return to the start and handle a zero remainder in the move count. A correct final digit without those reasons is not yet a complete explanation.

Review, repair and continue

Compact answer key

D1: The claimed remainder 7 is invalid; $43 = 6 \times 7 + 1$.

D2: Position 8: Yellow. Position 9: Red.

E1: Position 2 after 35 moves.

E2: 4^{20} leaves remainder 2; $4^{21} - 1$ is divisible by 7.

P1: 37, 45, 53.

P2: Position 7; reachable places 1, 3, 5, 7.

P3: Exactly remainders 0 and 2.

P4: 2, 5, 8, 11, 14; five exponents.

P5: 6, 12, 18, 24, 30.

P6: $n = 3q + 1$ or $3q + 2$ for integers $q \geq 0$.

X1: 0 after 100 moves.

Common mistakes and repairs

Accepting a leftover that is at least the divisor as a remainder. Regroup another full set and check $0 \leq r < d$. Use D1.

Treating remainder 0 as no answer, or always selecting the first entry of a position-1 list. Mark the starting index. In a move sequence, complete cycles return to the starting state. In D2, a position divisible by 4 is the final colour of a block.

Announcing a cycle from a few terms without explaining why the rest repeats. Point to the exact repeated state and the unchanged update rule. Then explain why each following state is forced to repeat.

Using only a few sample values to prove an impossible remainder. List every remainder of the input on division by the relevant divisor, as in P3.

Finding one working index when the question asks for all of them. State the relevant index remainder, list its values in the permitted interval and check the values immediately outside the interval.

Four checks for one written answer

• I stated the permitted input values and the relevant divisor or starting index.

• My remainder calculations are valid and every remainder is between 0 and one less than the divisor.

• I explained why my cycle repeats or why my cases cover every allowed input.

• I answered the whole question, checked the boundary or zero-remainder case when relevant, and can explain the answer after closing the solution.

Mark each criterion as shown, needs a repair or not relevant to this question. Then label the attempt independent, hint-supported or solution-assisted. Choose one repair before attempting a harder problem.

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